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THE VERTICAL DISTRIBUTION OF ATMOSPHERIC EDDY ENERGY

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INTRODUCTION

In this paper an attempt is made to study theoretically the vertical variation of atmospheric eddy viscosity (c_M) and its relations to the distribution of temperature and wind with height. An investigation of this kind seems desirable for several reasons. The observed values of C_M vary between such wide limits that it hardly is possible to say what the individual observations really represent.[1] A rational grouping and interpretation of these empirical data can not be accomplished until theoretical considerations have given an indication as to the connection between c_M and the chief factors which regulate the production and consumption of atmospheric eddies, viz, the vertical stability of the stratification and the rate of sheering between superposed layers of different horizontal velocity.

Recently a few systematic attempts have been made to calculate the vertical distribution of c_M from observations of the variation of wind with height, the horizontal barometric gradient being known (*1*). The curves thus obtained show certain characteristic features, which likewise seem to require a theoretical explanation.

The author has already on an earlier occasion (*2*) tried to show how a solution of the problem above formulated could be approached; continued study of it seemed however to indicate that the method then used would not be adequate to the purpose and it was abandoned. The chief aid in the paper here presented was obtained by adapting a certain theorem in an important paper by L. F. Richardson (*3*), giving an extension and application of Reynold's Criterion of Turbulence to atmospheric systems. This theorem has the form of an integral equation, expressing the rate of increase of the total energy within a vertical air column as the sum of the rates of inflow of eddies through the boundaries and production of eddies within the system. Richardson's theorem is here applied to air columns of infinitesimal height and thus changed into a differential equation. In performing this transformation it was necessary to make a hypothesis concerning the vertical transport of eddy energy. The one here made is that it follows the same law as the vertical transport of heat and momentum, in accordance with a suggestion made by Richardson (*3*).

I. THE FUNDAMENTAL EQUATION

Let x, y, and h be the three orthogonal coordinates (h the height above ground); let E' be the total turbulent energy within the atmospheric volume V and u and v the two horizontal mean wind components in the direction of x and y, respectively. Then the rate at which the kinetic energy of the mean motion is transformed into eddy energy can be expressed by the formula

$$(1)\qquad \iiint_V c_M\left[\left(\frac{\partial u}{\partial h}\right)^2+\left(\frac{\partial v}{\partial h}\right)^2\right]dx\,dy\,dh$$

During turbulent stirring of the air a certain part of the eddy energy is transformed into potential and thermal energy. The rate of this energy conversion is, according to Richardson:

$$(2)\qquad \iiint_V c_H\,\frac{g}{T}\,(\alpha_o-\alpha)\;dx\,dy\,dh$$

Here c_H represents the eddy convectivity, g the gravity acceleration, T the absolute temperature, α_o the adiabatic, and α the actual, vertical lapse rate of temperature. It is seen from (2) that in the case of superadiabatic lapse rate the process is reversed, i. e., potential and thermal energy are then transformed into turbulent energy.

Through the action of molecular viscosity a certain fraction of E' is transformed into heat. For this fraction Richardson gives the expression

$$(3)\qquad \nu E',$$

where the coefficient ν depends upon the linear dimensions of the eddies. If we assume, however, that their size remains constant throughout the portion considered, we may regard ν as a constant.

If throughout the boundary there is no transport of eddies to or from V, then the sum of the quantities (1), (2), and (3), each taken with its proper sign, must be equal to the total change in E'. Thus we get Richardson's equation

$$(4)\qquad \frac{\partial E'}{\partial t}=\iiint_V\left\{c_M\left[\left(\frac{\partial u}{\partial h}\right)^2+\left(\frac{\partial v}{\partial h}\right)^2\right]-c_H\,\frac{g}{T}\,(\alpha_o-\alpha)\right\}dx\,dy\,dh-\nu E'$$

Let us now apply (4) to an atmospheric system of unit cross section and height dh. The equation (4) then changes into

$$(5)\qquad \rho\,\frac{\partial E}{\partial t}\,dh=c_M\left[\left(\frac{\partial u}{\partial h}\right)^2+\left(\frac{\partial v}{\partial h}\right)^2\right]dh-c_H\,\frac{g}{T}\,(\alpha_o-\alpha)\;dh-\nu\rho E dh$$

Here E denotes the eddy energy per unit mass. To the second member of this equation there must be added an expression for the rate of inflow of eddy energy through the boundaries.

[1] A very complete table of observed values of c_M is given by L. F. Richardson in "Weather Prediction by Numerical Process," Cambridge, the University Press, 1922.

We know that turbulence gives rise to a vertical transfer of horizontal momentum with the components

$$-c_M \frac{\partial u}{\partial h},\ -c_M \frac{\partial v}{\partial h}$$

In the same way the eddy transfer of potential temperature (θ) can be expressed in the form:

$$-c_H \frac{\partial \theta}{\partial h}$$

Now, it is known from experience that eddies are able to diffuse from one layer to another; in fact, this is what we observe on a large scale when we watch the growth of the summertime cumulus clouds. We assume as an approximate expression for the rate of this diffusion of eddy energy:

$$-c_E \frac{\partial E}{\partial h},$$

which is analogous to the formulae for the eddy transfer of heat and momentum previously given. A similar expression has already been proposed by Richardson.[2]

Taking into account this vertical diffusion of eddy energy, we finally obtain:

$$\rho \frac{\partial E}{\partial t} = c_M\left[\left(\frac{\partial u}{\partial h}\right)^2 + \left(\frac{\partial v}{\partial h}\right)^2\right] - c_H \cdot \frac{g}{T}(\alpha_o - \alpha) - \nu\rho E + \frac{\partial}{\partial h}\left[c_E \frac{\partial E}{\partial h}\right] \tag{6}$$

As long as the relations between c_M, c_H, c_E, and E are unknown, the equation (6) is of little value. If c_M, c_H, and c_E could be expressed in terms of E, then the solution of (6), together with certain boundary conditions, ought to give the vertical distribution of eddy energy corresponding to given values of α and $\left(\frac{\partial u}{\partial h}\right)^2 + \left(\frac{\partial v}{\partial n}\right)^2$.

We know that in systems where the average size of the eddies remains constant the quantities c_M, c_H, and in all probability also c_E, increase with E. Furthermore, we know that they disappear for $E = O$. Consequently, as a first approximation, we may introduce the expressions:

$$\begin{aligned} c_E &= aE \\ c_M &= bE \\ c_H &= cE, \end{aligned} \tag{7}$$

where a, b, and c are constants. From observation it has been found[3] that

$$c_M = c_H \tag{8a}$$

Therefore,

$$b = c. \tag{8b}$$

In the numerical examples given later the identities (8a) and (8b) are extended to include also c_E and a, viz,

$$a = b = c. \tag{8c}$$

It must be kept clearly in mind that through this assumption real conditions are depicted only in their rough outlines. This becomes still plainer if we remember that c_M probably has two very different values for the transport of momentum parallel to the wind and across the wind (Richardson, "Weather Prediction by Numerical Process," page 73.)

Combining (6), (7), and (8), we obtain

$$\rho \frac{\partial E}{\partial t} = a \frac{\partial}{\partial h}\left[\frac{\partial E}{\partial h}\right] + E\left\{b\left[\left(\frac{\partial u}{\partial h}\right)^2 + \left(\frac{\partial v}{\partial h}\right)^2\right] - b \cdot \frac{g}{T}(\alpha_o - \alpha) - \nu\rho\right\} \tag{9}$$

From this equation it should be possible to derive the vertical distribution of E as well as the change of E with time, when the boundary conditions are given. If the equation (9) can be verified and the constants a, b, and ν properly determined, then it offers one great advantage for the determination of E. Computing c_M and thus also E in the ordinary way from the hydrodynamical equations, we obtain the formula

$$c_M(h_1) = \frac{c_M(h_2)\left(\frac{\partial u}{\partial h}\right)_{h_2} + 2\rho\Omega \sin\varphi \int_{h_1}^{h_2} v\,dh}{\left(\frac{\partial u}{\partial h}\right)_{h_1}}.$$

The x-axis runs in the direction of the gradient wind. Ω is the angular velocity of the earth and φ the latitude. c_M is therefore expressed as a ratio between two quantities which in a rather short distance from the ground rapidly approach zero. Thus a hydrodynamical determination of c_M for greater heights becomes almost impossible. This difficulty is avoided by use of the equation (9).

It is easily seen that on account of the diffusion term $a \frac{\partial}{\partial h}\left[E \frac{\partial E}{\partial h}\right]$ the equation (9) is no longer linear. This means, mathematically, that the sum of two partial solutions of (9) will not give a new solution. Attempts have been made to differentiate between "thermal" and "dynamical" turbulence, the latter being the turbulence which is supplied exclusively from the kinetic energy of the mean motion. It is easily seen from (9) that the two kinds of turbulence react upon each other. Attempts to differentiate between them must therefore be futile. However, if we assume an atmosphere in which the components u and v of the mean motion vanish, we may speak of a pure convective or thermal turbulence. In the same way it is possible to discuss a pure dynamical turbulence, if the study is limited to an incompressible liquid, for instance a turbulent river. In the following section some integrals of (9) for the limiting case of pure convective turbulence are given.

Before deriving these integrals we shall, however, draw one conclusion of a more general character from (9). Under stationary conditions this equation takes the form

$$a \frac{\delta}{\delta h}\left[E \frac{\delta E}{\delta h}\right] + \psi' E = O \tag{10}$$

where

$$\psi' = \left\{b\left[\left(\frac{\delta u}{\delta h}\right)^2 + \left(\frac{\delta v}{\delta h}\right)^2\right] - b \cdot \frac{g}{T}(\alpha_o - \alpha) - \nu\rho\right\} \tag{11}$$

Now integrate (10) between H_1 and H_2 and suppose that $E \frac{\delta E}{\delta h}$ is equal to zero at these limits. The result is:

$$\int_{H_1}^{H_2} \psi' \cdot E\,dh = O. \tag{12}$$

[2] Richardson's expression does not apply to the eddy energy per mass (E) but to the "potential eddy energy per mass," $E\left(\frac{\rho_i}{\rho}\right)^{\frac{2}{3}}$, referred to some standard density ρ_i Richardson arrives at this expression through the application of thermodynamical methods to the study of eddies. He proves that if a large air volume containing numerous eddies is expanded $E\left(\frac{\rho_i}{\rho}\right)^{\frac{2}{3}}$ remains constant. Thus the eddies behave like the molecules of a monatomic gas. For the schematic theory outlined in the following pages we shall, however, use the simpler diffusion expression given above.

[3] This was first suggested by Taylor (Phil. Transactions, London, 1915) and later confirmed by Richardson.

Since E is essentially positive, the integral (12) can not vanish unless ψ' changes its sign at least once between H_1 and H_2 or remains equal to zero in the entire interval. In layers where ψ' is positive, eddies are produced; where ψ' is negative, eddies are consumed. Since consumption and production of eddies are proportional to ψ', we may introduce for this quantity the name "eddy productivity." Thus: If within a volume V the state of turbulence is stationary and through the boundaries no transport of eddies takes place, then V must contain eddy producing as well as eddy consuming layers. In the atmosphere the eddy producing layers are generally found next to the ground, where the vertical increase of wind velocity and the temperature gradient have their greatest values. The upper layers, in which the wind remains more or less constant and the temperature lapse rate has a smaller value, are generally eddy consuming.

II. CONVECTIVE TURBULENCE IN THE FREE ATMOSPHERE

All through this section it will be assumed that the velocity components of mean motion (u, v) vanish. The eddy productivity ψ' therefore reduces to

$$\psi^1 = -\left\{b \cdot \frac{g}{T}(\alpha_o - \alpha) + \nu\rho\right\}.$$

As has been pointed out, the numerical value of ν largely depends on the size of the eddies. Assuming their average linear dimensions to be about 10 meters, Richardson finds for ν a value of $2.6.10^{-6}$ sec.$^{-1}$. This means that frictional forces alone would in 24 hours reduce a given supply of eddy energy to 0.8 of its original value. Now it is seen from the expression (I, 11) that the vertical stability of stratification acts upon the eddies in the same way as does the friction. Through the numerical examples treated below it will be shown that the smothering influence of the vertical stability is far more effective than that of the viscosity. Therefore, it seems justifiable to neglect in the expression for the eddy productivity the term $\nu\rho$, at least when the lapse rate is not too close to the adiabatic. Thus we get

$$\psi' = -b \cdot \frac{g}{T}(\alpha_o - \alpha) \tag{13}$$

Let us now suppose that in an unlimited atmosphere the following temperature distribution is maintained through radiation. In a certain layer, between $h = -H$ and $h = +H$, the lapse rate is constant and superadiabatic,

$$a > a_o.$$

In the adjacent layers the lapse rate is constant and less than adiabatic,

$$a_1 < a_o.$$

In the intermediate layer we have a positive eddy productivity,

$$\psi_o' = b \cdot \frac{g}{T}(a - a_o).$$

In the surrounding layers the eddy productivity is negative,

$$\psi_1' = -b \cdot \frac{g}{T}(a_o - a_1).$$

If the variations in T be neglected, then the quantities ψ_o' and ψ_1' may be considered as constants. We shall now try to compute a stationary distribution of eddy energy E, that corresponds to the given distribution of eddy productivity. Under stationary conditions the fundamental equation (I, 9) has the form:

$$a\frac{\delta}{\delta h}\left[E\frac{\delta E}{\delta h}\right] + E\psi' = O.$$

Let us first discuss the intermediate layer. There we have

$$\frac{\delta}{\delta h}\left[E\frac{\delta E}{\delta h}\right] + \psi_o E = O, \tag{14}$$

where

$$\psi_o = \frac{b}{a}\frac{g}{T_o}(a - a_o) \tag{15}$$

T_o is the absolute temperature at the center of this layer. The equation (14) can be written

$$\frac{\delta E}{\delta h} \cdot \frac{\delta}{\delta E}\left(E\frac{\delta E}{\delta h}\right) + \psi_o \cdot E = O,$$

or

$$2E\frac{\delta E}{\delta h} \cdot \frac{\delta}{\delta E}\left(E\frac{\delta E}{\delta h}\right) = -2\psi_o \cdot E^2$$

Integrating with respect to E, we obtain

$$E^2\left(\frac{\delta E}{\delta h}\right)^2 = C - \frac{2\psi_o}{3}E^3. \tag{16}$$

Since the variation of T with height has been neglected, it is obvious that E must be symmetrical with regard to h. Thus $\frac{\delta E}{\delta h}$ must vanish for $h = O$. If we denote by E_o the value of E at this height, we can write (16) in the form

$$E^2\left(\frac{\delta E}{\delta h}\right)^2 = \frac{2\psi_o}{3}(E_o^3 - E^3). \tag{16b}$$

Now let us introduce the quantity

$$\frac{E}{E_o} = z. \tag{17}$$

Then we get

$$z\frac{\delta z}{\delta h} = -\sqrt{\frac{2\psi_o}{3E_o}}\sqrt{1 - z^3}, \tag{18}$$

or, after integration,

$$\int_1^z \frac{z\,dz}{\sqrt{1 - z^3}} = -\sqrt{\frac{2\psi_o}{3E_o}}\,h. \tag{19}$$

In the surrounding layers we have

$$\frac{\delta}{\delta h}\left[E\frac{\delta E}{\delta h}\right] - \psi_1 E = O, \tag{20}$$

where

$$\psi_1 = \frac{b}{a}\frac{g}{t_0}(\alpha_0 - \alpha_1). \tag{21}$$

Treating this equation in the same way as (14), we obtain

$$2E\frac{\delta E}{\delta h}\frac{\delta}{\delta E}\left(E\frac{\delta E}{\delta h}\right) = 2\psi_1 E^2$$

or, after integration,

$$E^2\left(\frac{\delta E}{\delta h}\right)^2 = K + \frac{2\psi_1}{3}E^3. \tag{22}$$

The eddies which are generated in the intermediate layer and diffuse upward and downward will gradually be consumed in the surrounding layers and their energy will be transformed into potential and thermal energy. At a certain distance the dissipation of the eddy energy will be complete, i. e., we there have $E=O$. Since $\frac{\delta E}{\delta h}$ in any case can not be infinite at this height, it follows that

$$K=O.$$

The equation (22) therefore is simplified to

$$\left(\frac{\delta E}{\delta h}\right)^2=\frac{2\psi_1}{3}\cdot\ E. \tag{23}$$

This gives, after integration,

$$E=\frac{\psi_1}{6}\cdot\ (H_1-h)^2. \tag{24}$$

H_1 is the constant of integration and gives the height at which the eddy energy is totally dissipated.

Let us next discuss the conditions at the boundaries between the layers. There the following requirements must be fulfilled:

(1) The diffusion stream of eddy energy must be continuous.

(2) The eddy energy E must be continuous.

If we denote by E_1 the value of E for $h=H$, then we obtain from these conditions and from the equations (16b) and (23),

$$\frac{2\psi_1}{3}E_1^3=\frac{2\psi_O}{3}(E^3_O-E_1^3), \tag{25}$$

or

$$\frac{E_1}{E_O}=\sqrt[3]{\frac{\psi_O}{\psi_O+\psi_1}}=\sigma(<1). \tag{26}$$

Combining (26) and (19), we obtain

$$-q=\int_1^\sigma\frac{z\,dz}{\sqrt{1-z^3}}=-\sqrt{\frac{2\psi_O}{3E_O}}\cdot\ H. \tag{27}$$

Since σ is a known quantity, q can be computed and we are therefore able to determine E_O from the equation (27). The result is

$$E_O=\frac{2\psi_O}{3\,q^2}\cdot\ H^2 \tag{28}$$

We still have to compute the constant H_1. From (24) it follows that

$$\sqrt{\frac{6E_O\sigma}{\psi_1}}+H=H_1 \tag{29}$$

or

$$H_1=H\left(1+\frac{2}{q}\sqrt[6]{\frac{\psi_o^4}{\psi_1^3(\psi_o+\psi_1)}}\right). \tag{30}$$

From the formulæ (28) and (30) it is seen that the maximum intensity of the eddy energy is proportional to the square of the height of the eddy-producing layer. The total height of the turbulent layer is proportional to the height of the eddy-producing layer. In the table below the solution of the problem is given in a condensed form.

(A) between $h=O$ and $h=H$

$$\sigma=\sqrt[3]{\frac{\psi_o}{\psi_o+\psi_1}} \qquad E=zE_o$$

$$q=\int_\sigma^1\frac{z\,dz}{\sqrt{1-z^3}}$$

$$\varphi(z)=\int_z^1\frac{z\,dz}{\sqrt{1-z^3}}=\sqrt{\frac{2\psi_o}{3E_o}}\,h$$

$$E_o=\frac{2\psi_o}{3q^2}H^2$$

(B) between $h=H$ and $h=H_1$

$$H_1=H\left(1+\frac{2}{q}\sqrt[6]{\frac{\psi_o^4}{\psi_1^3(\psi_o+\psi_1)}}\right)$$

$$E=\frac{\psi_1}{6}(H_1-h)^2.$$

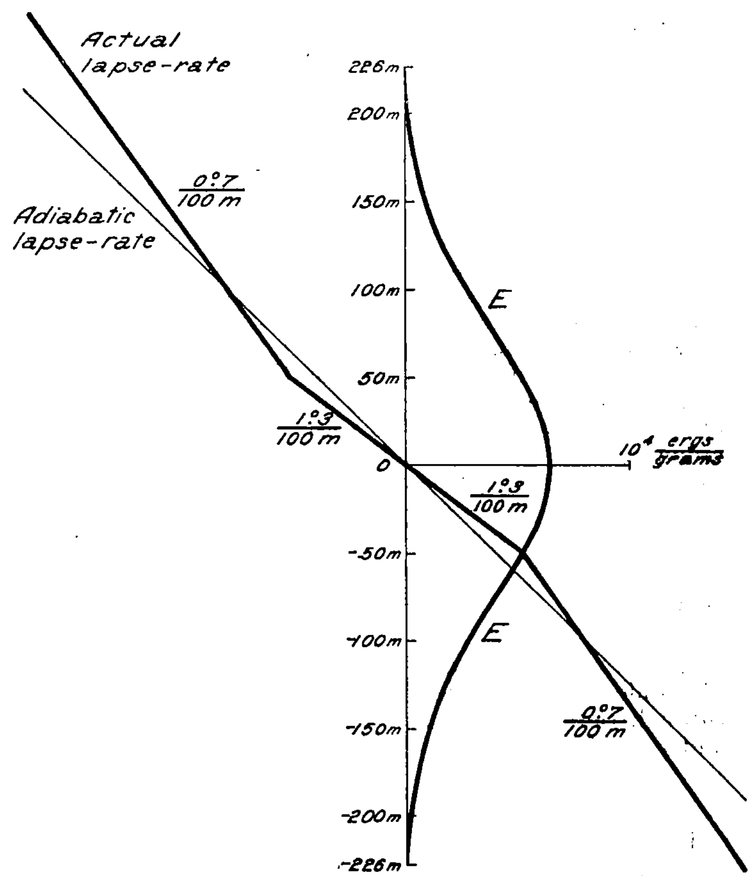


FIG. 1

As an illustration of this integral a numerical example has been computed and plotted in Figure 1. The thickness 2H of the superadiabatic layer has been assumed to be 100 m., the superadiabatic lapse rate α 1.3° C. per 100 m., and the lapse rate in the surrounding layers 0.7° C. It is seen from the diagram that the height at which the eddy current from the superadiabatic layer totally disappears, is equal to 226 m. This value is independent of the numerical value of the ratio $\frac{b}{a}$, and should therefore be rather reliable. The maximum value of the eddy energy is equal to 0.64. $10^4\ \frac{\text{ergs}}{\text{gram}}$.

In Figure 2 a diagram is given of the elliptic integral

$$\varphi(z)=\int_z^1\frac{z\,dz}{\sqrt{1-z^3}},$$

which occurs in the solution given above.

It may seem absurd and in contradiction with observed conditions that a superadiabatic layer like the one here assumed should not exert its effects through more than 226 m. in each direction, especially if we compare this result with conditions on a clear, warm summer afternoon, when the convection currents from a rather thin superadiabatic layer at the surface extend thousands of meters in height. We must, however, keep in mind, that we have assumed the eddy producing layer to be surrounded by stable air masses with a lapse rate of 0.7° C. per 100 m., while in the case of summer convection the superadiabatic gradient in the bottom layer is followed above by an adiabatic lapse rate extending practically as far as the convection currents themselves. Therefore, the numerical example just treated illustrates how effective even the comparatively high gradient of 0.7° C. is in suppressing turbulence. Thus we may conclude, that the eddies created in the lower layers of a stable air mass as soon as there is wind, are able to diffuse upward only with great difficulty. In this respect the stable atmosphere differs widely from an homogeneous incompressible fluid, for instance a river, where only frictional forces counteract the upward diffusion of the turbulence from the bottom layers.

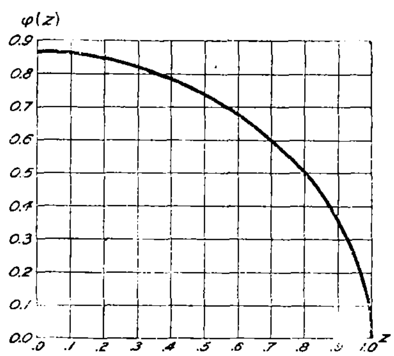


FIG. 2

Returning to the fundamental equation (I, 9) we shall now compute how an initial, limited eddy supply is diffused upward and downward under different temperature conditions. Assume the atmosphere to be unlimited and the vertical lapse rate to be $\alpha<\alpha_o$. The equation then takes the form

$$\frac{\rho}{a}\,\frac{\delta E}{\delta t}=\frac{\delta}{\delta h}\left[E\,\frac{\delta E}{\delta h}\right]-\psi E, \tag{31}$$

where

$$\psi=\frac{b}{a}\,\frac{g}{T}\,(\alpha_o-\alpha). \tag{32}$$

Here T is regarded as a constant. If we introduce new variables,

$$\tau=\frac{\alpha\psi t}{\rho} \tag{33}$$

and

$$x=h\sqrt{\psi}, \tag{34}$$

then the equation (31) can be simplified to

$$\frac{\partial E}{\partial\tau}=\frac{\partial}{\partial x}\left[E\frac{\partial E}{\partial x}\right]-E. \tag{35}$$

Suppose the eddy disturbance E to be limited to a region

$$-\sqrt{\frac{l}{m}}<x<+\sqrt{\frac{l}{m}}$$

and assume for E the analytical form

$$E=l-mx^2. \tag{36}$$

In this expression l and m are functions of τ. Substituting this expression for E in (35), we find

$$(l'+l)-(m'+m)x^2+\frac{\partial}{\partial x}\,(2lmx-2m^2x^3)=O,$$

or

$$(l'+l)-(m'+m)x^2+2lm-6m^2x^2=O. \tag{37}$$

The primes indicate differentiation with respect to τ. From (37) it is evident that the following requirements must be fulfilled, namely:

$$\begin{aligned} l'+l+2lm&=O\\ m'+m+6m^2&=O. \end{aligned} \tag{38}$$

The second equation (38) gives after integration

$$m=\frac{m_oe^{-\tau}}{6m_o+1-6m_oe^{-\tau}} \tag{39}$$

and the first

$$l=\frac{l_oe^{-\tau}}{\sqrt[3]{6m_o+1-6m_oe^{-\tau}}}. \tag{40}$$

Here l_o and m_o are the values of l and m for the time $\tau=O$. The upper limit (H) for the turbulent layer is given by the equation:

$$H^2=\frac{l}{m\psi}$$

or

$$H^2=\frac{l}{m_o\psi}\,[(6m_o+1)-6m_oe^{-T}]^{\frac{2}{3}}.$$

Taking the square root of each member, we obtain

$$2H=2H_o\sqrt[3]{6m_o+1-6m_oe^{-\tau}}. \tag{41}$$

$2H$ is the thickness of the turbulent layer at the time τ and $2H_o$ the thickness of the same layer at the time O. We see that the velocity of the diffusion decreases more and more. The layers outside of

$$\pm H_\infty=\pm H_o\sqrt[3]{6m_o+1} \tag{42}$$

will remain unaffected by the eddies.

The total amount of eddy energy, given by the expression

$$K=\int_{-H}^{+H}E\,dh$$

is equal to

$$K=\frac{4}{3}lH.$$

It is easily found that this expression has the value

$$K=K_oe^{-\frac{\alpha\psi t}{\rho}} \tag{43}$$

K_o means the total eddy energy at the time $t=O$. The total turbulent energy decreases more and more; at the time $t=\infty$, when the disturbance has reached the limits $\pm H_\infty$, all the eddy energy has been consumed and transformed into potential and thermal energy.

Now suppose that the lapse rate has such a value that the eddy productivity disappears, i. e., let us assume

$$b\,\frac{g}{T}\,(\alpha_o-\alpha)+\nu\rho=O.$$

The diffusion equation then reduces to

$$\frac{\partial E}{\partial\tau}=\frac{\partial}{\partial h}\left[E\,\frac{\partial E}{\partial h}\right] \tag{44}$$

where

$$\tau=\frac{at}{\rho}. \tag{45}$$

Finally consider the case of unstable stratification, where $\alpha>\alpha_o$.

The diffusion equation then has the form

$$\frac{\partial E}{\partial\tau}=\frac{\partial}{\partial x}\left[E\,\frac{\partial E}{\partial x}\right]+E, \tag{47}$$

where

$$\tau=\frac{a\psi t}{\rho}$$

$$x=h\sqrt{\psi} \tag{48}$$

$$\psi=\frac{b}{a}\,\frac{g}{T}\,(\alpha-\alpha_o)$$

The integral is

$$E=l-m\psi h^2$$

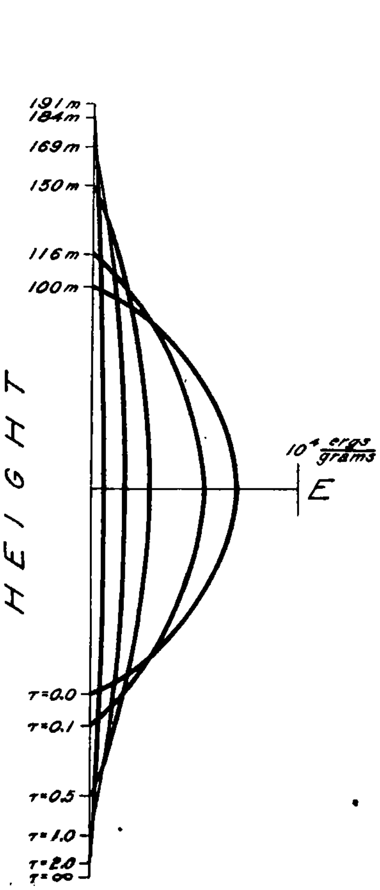


FIG. 3a

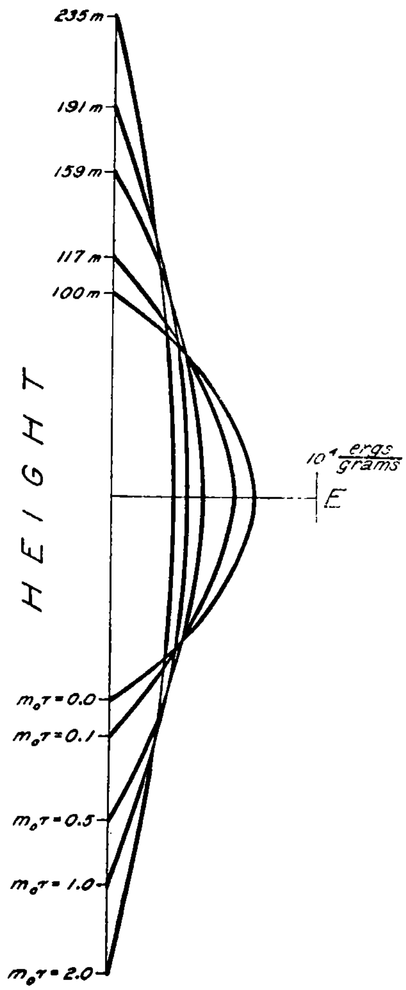


FIG. 3b

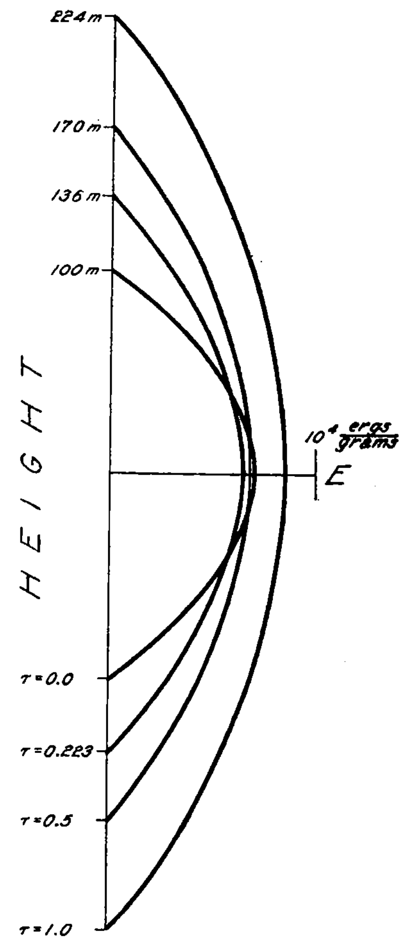


FIG. 3c

In this case the corresponding solution takes the form:

$$E=l-mh^2 \tag{46}$$

$$l=\frac{l_o}{\sqrt[3]{1+6m_o\tau}}$$

$$m_o=\frac{m_o}{1+6m_o\tau}$$

$$H=H_o\sqrt[3]{1+6m_o\tau}$$

It is seen that the disturbance now will travel toward the infinite, but with decreasing velocity. The total eddy energy in this case remains constant.

$$l=\frac{l_oe^\tau}{\sqrt[3]{6m_oe^\tau+(1-6m_o)}} \tag{49}$$

$$m=\frac{m_oe^\tau}{6m_oe^\tau+1-6m_o}$$

$$H=H_o\sqrt[3]{6m_oe^\tau+(1-6m_o)}.$$

We see that the disturbance travels toward infinity with an increasing velocity. The total eddy energy (K) increases at the rate

$$K=K_oe^{\frac{a\psi t}{\rho}},$$

the potential energy of the unstable stratification rapidly becoming transformed into eddy energy.

In Figures 3a, 3b, and 3c are given graphical representations in (E, h)-diagrams of this process of eddy diffusion during stable, neutral, and unstable stratifications.

In all three cases the initial disturbance is supposed to have the same form and magnitude; its maximum value for $h=O$ is $7 \cdot 10^3 \frac{\text{ergs}}{\text{gram}}$. For the stable lapse rate we have assumed the value 0.8° C./100 m. and for the unstable 1.2° C./100 m. Thus, if we put

$$\frac{b}{a}=1, \quad \frac{g}{T_o}=3.5$$

we get, in both cases,

$$\tau=7 \cdot 10^{-5} \frac{a}{\rho} t.$$

During neutral stratification the corresponding equation is

$$\tau=\frac{a}{\rho} t.$$

However, in that case m_o has the value $7 \cdot 10^{-5}$ and we obtain

$$m_o \tau=7 \cdot 10^{-5} \frac{a}{\rho} t.$$

Therefore, as long as a can be assumed to have the same value in all three cases, corresponding values of τ and $m_o\tau$ in figures 3a, 3b, and 3c ought to represent equal time intervals.

It must be pointed out that especially in the case of unstable stratification the theory above given is rather unsatisfactory. The steadily increasing turbulence will constantly tend to diminish the lapse rate to the adiabatic and thus prevent unlimited increase of E.

In the examples already treated the temperature distribution has been regarded as constant with time. It is well known that the eddy transfer of heat (eddy convection) is one of the chief factors determining the vertical temperature distribution, which therefore, contrary to the assumption hitherto made, generally varies with changes in the eddy distribution. If the potential temperature be denoted by θ, then the equation for heat convection by means of eddies has the form

$$\rho \frac{\partial \theta}{\partial t}=\frac{\partial}{\partial h}\left[c_H \frac{\partial \theta}{\partial h}\right] \tag{50}$$

Substituting from (I, 7), the expression for C_H, we obtain

$$\rho \frac{\partial \theta}{\partial t}=c \frac{\partial}{\partial h}\left[E \frac{\partial \theta}{\partial h}\right]. \tag{51}$$

In the general convection problem this equation must be solved simultaneously with (I, 9). If the quantities T and α, which occur in (I, 9) are expressed in terms of θ, it is easily seen that

$$\frac{1}{T}(\alpha_o-\alpha)=\frac{1}{T}\left(\alpha_o+\frac{\partial T}{\partial h}\right)=\frac{1}{\theta} \frac{\partial \theta}{\partial h}. \tag{52}$$

Thus the equations, which determine atmospheric convection, are

$$\rho \frac{\partial E}{\partial t}=a \frac{\partial}{\partial h}\left[E \frac{\partial E}{\partial h}\right]-c \cdot g \cdot \frac{E}{\theta} \frac{\partial \theta}{\partial h} \tag{53a}$$

$$\rho \frac{\partial \theta}{\partial t}=c \cdot \frac{\partial}{\partial h}\left[E \frac{\partial \theta}{\partial h}\right] \tag{53b}$$

General solutions of this system are not easily obtained. If, however, we assume $a=c$, and substitute a constant instead of θ in the last term of (53a), then one simple integral can be obtained, which may have some bearing upon the rise of cumuli on summer days. Introducing the new variable

$$\tau=\frac{at}{\rho}=\frac{ct}{\rho}$$

and putting

$$\frac{g}{\theta}=q \text{ (constant)},$$

we may write our equations in the form

$$\frac{\partial E}{\partial \tau}=\frac{\partial}{\partial h}\left[E \frac{\partial E}{\partial h}\right]-qE \cdot \frac{\partial \theta}{\partial h} \tag{54a}$$

$$\frac{\partial \theta}{\partial \tau}=\frac{\partial}{\partial h}\left[E \frac{\partial \theta}{\partial h}\right] \tag{54b}$$

Now suppose that in the layer $-H<h<+H$ the temperature gradient is superadiabatic, i. e., the potential temperature falling,

$$\theta=\theta_O-\alpha h \; (\alpha>O). \tag{55}$$

In the same layer the eddy energy is assumed to be distributed according to the law

$$E=k-lh^2=l(H^2-h^2) \tag{56}$$

$$\frac{k}{l}=H^2 \tag{57}$$

For $h>+H$ the temperature gradient is adiabatic, i. e., the potential temperature θ_1 constant,

$$\theta_1=\theta_O-\Delta.$$

For $h<-H$ we have another constant potential temperature

$$\theta_2=\theta_O+\Delta$$

The eddy energy is zero for $h>+H$ and $h<-H$.

The quantities α, k, and l are functions of τ, which must be so determined that the expressions (55) and (56) for E and θ satisfy the equations (54).

From (54b) and (55) we obtain

$$-\alpha' h=+2hl\alpha \qquad \left(f'=\frac{\partial f}{\partial \tau}\right)$$

or

$$\alpha'=-2\alpha l. \tag{58}$$

In the same way we obtain from (54a), (55) and (56)

$$k'-l'h^2-\alpha q(k-lh^2)+2lk-6l^2h^2=O$$

or

$$k'-\alpha qk+2lk-h^2(l'-\alpha ql+6l^2)=O.$$

Since this equation must be fulfilled for any value of h, we obviously must have

$$k'-\alpha qk+2lk=O. \tag{59}$$

$$l'-\alpha lq+6l^2=O. \tag{60}$$

Eliminating l between (58) and (60), we obtain, after integration,

$$\alpha = \frac{\alpha_o}{1 + \frac{\alpha_o q \tau}{2}}, \tag{61}$$

where α_o is the value of α for the time $\tau = O$. In the same way we get

$$l = \frac{\alpha_o q}{4\left(1 + \frac{\alpha_o q \tau}{2}\right)} \tag{62}$$

and, from (59), (61), and (62),

$$k = C\left(1 + \frac{\alpha_o q}{2}\tau\right), \tag{63}$$

where C is a constant of integration. This constant can be determined in the following way. We have

$$\theta_1 = \theta_o - \Delta = \theta_o - \alpha H$$

or

$$\Delta^2 = \alpha^2 H^2,$$

and, according to (57)

$$H^2 = \frac{k}{l}.$$

Thus

$$\Delta^2 = \alpha^2 \frac{k}{l},$$

or

$$\Delta^2 = \frac{\alpha_o^2}{\left(1 + \frac{\alpha_o q \tau}{2}\right)^2} \cdot \frac{C\left(1 + \frac{\alpha_o q \tau}{2}\right) \cdot 4 \cdot \left(1 + \frac{\alpha_o q \tau}{2}\right)}{\alpha_o q} = \frac{4 C \alpha_o}{q}.$$

Therefore

$$C = \frac{q \Delta^2}{4 \alpha_o}. \tag{64}$$

For H we obtain

$$H = \sqrt{\frac{k}{l}} = \frac{\Delta}{\alpha_o}\left(1 + \frac{\alpha_o q \tau}{2}\right). \tag{65}$$

The solution may be written in the following condensed form:

$$\begin{aligned} &\sigma = \frac{q \alpha_o}{2}; \quad \alpha_o H_o = \Delta \\ &H = H_o (1 + \sigma \tau) \\ &\alpha = \frac{\alpha_o}{1 + \sigma \tau} = \frac{\Delta}{H} \\ &\theta = \theta_o - \frac{\Delta}{H} h \\ &E = \frac{q \Delta}{4H} (H^2 - h^2). \end{aligned} \tag{66}$$

The constant velocity of diffusion is equal to

$$H_o \sigma \cdot \frac{a}{\rho} = \frac{q \Delta}{2} \frac{a}{\rho} \; cm/sec. \tag{67}$$

If in both the surrounding layers the lapse rate is less than adiabatic, then the velocity of diffusion obviously has a smaller value than that computed above.

In Figure 4 a graphical illustration of the solution (66) is given. At the time $\tau = O$ the superadiabatic layer has a thickness of 200 m.; the difference in potential temperature between the upper and the lower air masses is equal to 6° C. The distribution of eddy energy at different times is given by parabolae. The potential temperature of the turbulent layer is represented by a bundle of straight lines through the point O. It is seen from the figure, how the eddy layer spreads, while at the same time the superadiabatic lapse rate more and more decreases.

The difference in potential temperature between the upper and lower air layers in the numerical example just given may seem rather great. It is therefore appropriate to show that such differences really occur in the free atmosphere. The kite ascent at Mount Weather for October 2, 1913 (*4*) gives between 2,711 m. and 4,102 m., a temperature lapse rate of about 1.5° C. per 100 m. and in the interval 3,020 m.—3,730 m. even 1.7° C. This corresponds to an upward decrease in potential temperature of about the same magnitude as that assumed above. The reason why in this case the superadiabatic lapse rate can be maintained long enough to make it possible for us to measure it with our instruments may probably be sought in the very stable stratification of the lower air layers (between 1,999 m. and 2,673 m. the ascent gives a lapse rate of 0.6° C. per 100 m.), through which disturbances from the ground are prevented from diffusing upward.

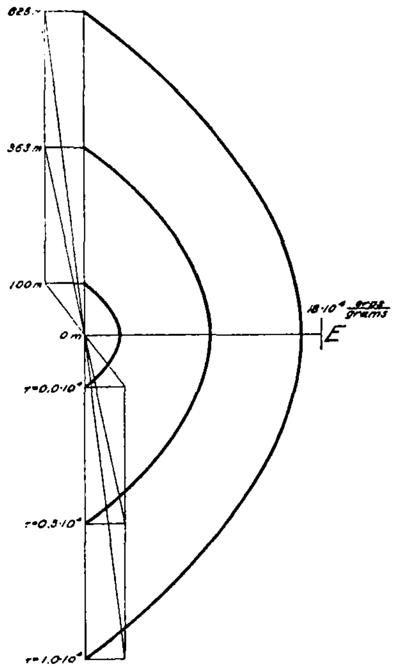


FIG. 4

The solution just derived may be applied to the discussion of the cause of certain thunderstorms. It has been observed, especially in higher latitudes, that thunderstorms frequently occur in the south quadrant of barometric depressions. This is explained by Humphreys (*5*) in the following way: The movement of the upper atmospheric strata follows the isobars, i. e., in the south quadrant of a depression the wind aloft is westerly. The surface wind is—on account of friction against the ground—deviated toward the center of the depression. Thus, in the south quadrant we have a southwesterly surface wind which, coming from lower latitudes, generally brings air masses which potentially are warmer than the masses transported with the upper west wind. In this way an unstable temperature distribution is created, which to a certain extent corresponds to the one here assumed, and a more or less violent overturning of the air sets in. In an unpublished paper by Mr. H. L. Choate, of the United States Weather Bureau, similar considerations have been applied as a more general explanation of cyclonic rains over the United States.[1]

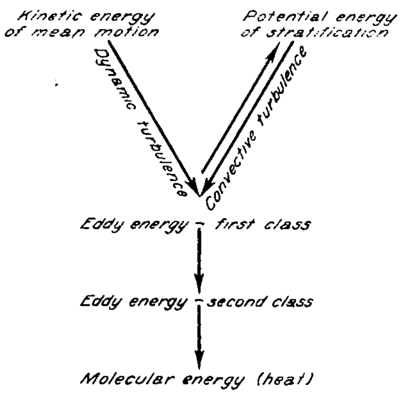
III. BOUNDARY CONDITION

Observations show that within the lowest air layers the atmospheric eddy viscosity c_M very rapidly decreases the more we approach the ground. Since we have assumed

$$c_M = aE, \tag{1}$$

where a is a constant, this should indicate that E, also, decreases with decreasing height above the surface. However, the coefficient a is a constant only as long as the average size of the eddies remains the same. Now the eddies in the surface layer are very small. They increase in size with distance from the ground. The above conclusion is therefore not justified.

Taking into account the variation of the size of the eddies, a should no longer be regarded as a constant but as a function of the height (h). In the same way the coefficients b, c, and ν should be replaced by functions of h. In order to avoid the difficulties arising from these more general expressions for a, b, c, and ν we may use the following method: Instead of regarding the dimensions of the eddies as continuously varying quantities, we will assume that all eddies can be grouped into two distinctly separate classes of entirely different magnitudes. In the first class we count all the large eddies of the free atmosphere. Only eddies belonging to this class are of significance for atmospheric eddy convectivity and viscosity. The eddies of the second class are smaller; in the same way as we neglect molecular heat conduction and friction compared with the eddy transfer of heat and momentum, we will also neglect the transfer of heat and mean momentum through eddies of the second class compared with the corresponding transport through eddies of the first class. It is well known that large eddies generally disintegrate into smaller and smaller eddies, which rapidly disappear under the action of molecular viscosity. Thus we obtain the scheme for energy transformations in the atmosphere, illustrated above.



We shall here make an additional assumption, namely, that the amount of eddy energy of the second class, which is formed directly from the kinetic energy of the mean motion or from the potential energy of stratification, must be small compared with the amount of eddy energy of the first class produced during the same time.

It is probable that sometimes several small eddies join to form a large eddy, but since we know nothing about this process it will not be considered here.

Under the assumptions made above, the equation (I, 6) can be used unaltered as an energy equation for eddies of the first class. E now means not the total eddy energy per unit mass but the eddy energy of the first class. νE gives the percentage of E, which per unit time is transformed into eddy energy of the second class. The interpretation of the other terms is obvious.

[1] This paper was presented at the April, 1924, meeting of the American Meteorological Society.

Now let us return to the boundary conditions. If only eddies of the first class are considered, then the relation (1), where a is a constant, may still be supposed to be fulfilled. Since c_M very rapidly goes toward zero when we approach the ground, it must be concluded that the value of E for $h=O$ is equal to zero. Now we will assume that in the same way as the kinetic energy of the mean motion is very rapidly converted into eddies at the surface and in the lowest layer, where the rate of sheering is strongest, in the same way eddies of the first class are dissolved into smaller eddies along the ground on account of friction at the surface. This loss (M) of eddy energy of the first class at the ground must obviously be equal to the downward eddy diffusion current at the surface, thus

$$M = \left(aE\frac{\partial E}{\partial h}\right)_{h=O} \tag{2}$$

Since E is equal to zero for $h=O$, this means that $\left(\frac{\partial E}{\partial h}\right)_{h=O}$ must be infinite. Hence we get in the lowest layer a development for E of the form

$$\frac{a}{2}\frac{\partial E^2}{\partial h} = M + M_2h + M_3h^2 + \cdots$$

or

$$E = \sqrt{\frac{2M}{a}}\sqrt{h}\ (1 + \text{terms of higher order}) \tag{3}$$

IV. STATIONARY DISTRIBUTION OF E

The above obtained boundary condition may now be used for determining the vertical distribution of E under stationary conditions. For that purpose we must return to the equation

$$\frac{\delta}{\delta h}\left[E\frac{\delta E}{\delta h}\right] + \psi E = O, \tag{1}$$

where

$$\psi = \frac{b}{a}\left[\left(\frac{\delta u}{\delta h}\right)^2 + \left(\frac{\delta v}{\delta h}\right)^2 - \frac{g}{T}(\alpha_o - \alpha) - \frac{\nu\rho}{b}\right]. \tag{2}$$

The eddy productivity generally varies with the height (h), in which case the integration of (1) offers great difficulties. However, we may then divide the air column into a number of layers and to each of them attribute a suitable constant value of ψ, thus simplifying the integration.

Now integrate (1) in the following case. Between the ground ($h=O$) and the height H the eddy productivity ψ is constant, positive and equal to ψ_o. For $h > H$ the eddy productivity is equal to another, negative, constant, $-\psi_1$. Thus we have to integrate the equation

$$\frac{\delta}{\delta h}\left[E\frac{\delta E}{\delta h}\right] + \psi_o E = O. \tag{3}$$

between the limits O and H.

We obtain, as previously shown,

$$E^2\left(\frac{\delta E}{\delta h}\right)^2 = \text{constant} - \frac{2\psi_o}{3}E^3. \tag{4}$$

At the ground $E=O$. The constant of integration obviously must be positive, since it is equal to the square of the loss of eddy energy at the surface. From (4) it is seen, that E increases more and more until it reaches a maximum value E_m at the height H_m. At this height

$$\frac{\delta E}{\delta h} = O$$

and therefore

$$\text{Constant} = \frac{2\psi_o E_m^3}{3}.$$

Thus,

(5) $$E^2\left(\frac{\delta E}{\delta h}\right)^2 = \frac{2\psi_o}{3}[E_m^3 - E^3]$$

Substituting for E the expression

(6) $$E = zE_m$$

we obtain

$$z^2\left(\frac{\delta z}{\delta h}\right)^2 = \frac{2\psi_o}{3E_m}(1-z^3)$$

or, after integration,

(7) $$\int_0^z \frac{z\,dz}{\sqrt{1-z^3}} = \sqrt{\frac{2\psi_o}{3E_m}} \cdot h \quad (h < H_m).$$

Between the constants E_m and H_m we have the relation

(8) $$q = \int_0^1 \frac{z\delta z}{\sqrt{1-z^3}} = \sqrt{\frac{2\psi_o}{3E_m}} \cdot H_m$$

or

(9) $$E_m = \frac{2\psi_o}{3q^2} \cdot H_m^2$$

Between H_m and H we have

(10) $$\int_z^1 \frac{z\,dz}{\sqrt{1-z^3}} = \sqrt{\frac{2\psi_o}{3E_m}}(h - H_m).$$

If we denote by E_1 the value of E for $h = H$, then

(11) $$q_1 = \int_{z_1}^1 \frac{z\,dz}{\sqrt{1-z^3}} = \sqrt{\frac{2\psi_o}{3E_m}}(H - H_m) \qquad \left(z_1 = \frac{E_1}{E_m}\right).$$

In the upper eddy consuming layer we have

(12) $$\frac{\delta}{\delta h}\left[E\frac{\delta E}{\delta h}\right] - \psi_1 E = O,$$

and, after integration,

(13) $$E = \frac{\psi_1}{6}(H_2 - h)^2,$$

where H_2 is the constant of integration. E and $\frac{\delta E}{\delta h}$ being continuous at the boundary between the two layers, we obtain

$$\psi_1 E_1^3 = \psi_o[E_m^3 - E_1^3]$$

or

(14) $$z_1 = \frac{E_1}{E_m} = \sqrt[3]{\frac{\psi_o}{\psi_o + \psi_1}}.$$

From

(15) $$E_1 = \frac{\psi_1}{6}[H_2 - H]^2$$

and (14) it follows that

(16) $$H_2 = H + \frac{2H}{q+q_1}\sqrt[6]{\frac{\psi_o^4}{(\psi_o+\psi_1)\psi_1^3}}.$$

From (9) and (15) it follows that

(17) $$H_m = \frac{q}{q+q_1} \cdot H.$$

The amount of eddy energy, which per unit time is dissipated at the ground, is equal to

(18) $$\left(aE\frac{\delta E}{\delta h}\right)_{h=o} = \frac{4a\psi_o^2}{q} \cdot \frac{H^3}{(q+q_1)^3}.$$

From this formula it is seen that the dissipation of eddy energy at the surface is independent of the nature of the ground. That may be approximately true in the case of pure convection but is doubtless wrong as soon as wind is blowing and the turbulence is partly of mechanical origin. This becomes obvious if we compare the values of eddy viscosity over a rough land surface and over sea.

However, the influence of the ground will make itself apparent in an indirect way. If we assume that the wind vector ($\mathfrak{v}$) does not vanish at the surface, then we get the following boundary condition in determining the mean wind distribution:

(19) $$\mathfrak{R} = c_m \cdot \left(\frac{\partial \mathfrak{v}}{\partial h}\right)_{h=o}.$$

The vector $\mathfrak{R}$, the tangential force between the wind and the ground, is a function of the surface wind $\mathfrak{v}_{h=o}$ and has probably the form

(20) $$|\mathfrak{R}| = \text{constant} \cdot |\mathfrak{v}_{h=o}|^2$$

The constant in (20) gives a measure of the roughness of the surface. The two conditions (19) and (20) have the following consequence: Over a rough surface there will be very little slipping and therefore a rapid increase of wind velocity with height. On the contrary, a smooth surface will admit a high degree of slipping. Thus the increase of wind with height will in this case be much less pronounced and restricted to a rather shallow layer. The eddy productivity, being proportional to the square of the increase of wind velocity with height, therefore has a high value over a continent but remains—for the same gradient wind—comparatively small over a smooth surface, for instance, the sea.

A numerical example has been computed and plotted in Figure 5, illustrating the integral obtained above. The height of the eddy producing layer has been chosen as 200 m. Assuming within this layer an average vertical increase of wind velocity of 2 m/sec per 100 m. and a lapse rate of 0.6° C. per 100 m., we obtain for the eddy productivity the value

$$\psi_o = 4 \cdot 10^{-4} - \frac{g}{T} \cdot 0.4 \cdot 10^{-4} \sim 2.6 \cdot 10^{-4}.$$

In the upper eddy consuming layer the wind velocity is constant. Thus we get

$$\psi_1 = \frac{g}{T} \cdot 0.4 \cdot 10^{-4} \sim 1.4 \cdot 10^{-4}.$$

From these numerical data we compute a maximum value of E, $4.3 \cdot 10^4 \ \frac{\text{ergs}}{\text{gram}}$, at the height 135 m. The eddy energy totally disappears 599 m. above ground.

Now let us compare our computed E curve with the observed conditions. The eddy energy (E) itself can not be directly measured, but it is possible from observations of the gustiness of wind to get a rough idea of the magnitude of E.

M. Robitzsch has given a table showing the most frequent gustiness amplitudes corresponding to different mean wind velocities (*6*). In the numerical example just given we have assumed a total increase in wind velocity from the ground upward of 4 m. p. s. Since the gradient velocity generally can be assumed to be twice the surface velocity, the latter would in this case be 4 m. p. s. For this average wind the gustiness amplitude is about 3.7 m. p. s. According to Robitzsch about 25 extreme values of the horizontal wind velocity occur per minute. This would correspond to a period of about 5 sec. We may then substitute for the real, turbulent wind a periodically changing horizontal velocity,

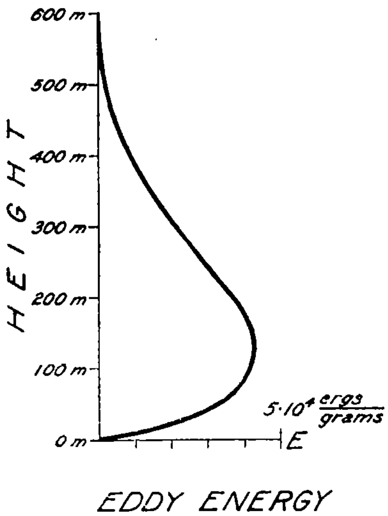


FIG. 5

$$v=\left(400+\frac{370}{2}\sin\frac{2\pi t}{5}\right)\frac{\text{cm}}{\text{sec}}.$$

If the vertical components of the eddy velocities are supposed to follow the same law, then the total eddy energy per unit mass must be equal to

$$E=\frac{3}{2}\cdot\frac{1}{2}\left(\frac{370}{2}\right)^2\cdot\frac{1}{5}\int_0^5\sin^2\frac{2\pi t}{5}\,dt=\frac{3}{160}(370)^2=2.6\cdot10^3\ \frac{\text{ergs}}{\text{gram}}.$$

Now, Robitzsch's data are based upon records from an instrument placed about 7 m. above ground. Our theoretical solution gives at the same height an E-value several times larger, $12.9\cdot10^3\ \frac{\text{ergs}}{\text{gram}}$. It must, however, be kept in mind, that the vertical change in E at these lower levels is so great that a comparison between theoretical and observed values for single points becomes extremely difficult.

Since, according to (I, 7), E and c_M are proportional, the computed curve also represents the variation of the eddy viscosity with height. It is seen, that in its general features this curve well agrees with Solberg's result and also with some curves for the vertical variation of c_M over the sea, which the author has calculated from certain pilot balloon observations[4] over the North Atlantic during the summers 1924 and 1925. (Fig. 6.) The curve marked *s* is made up from winds with a southerly component and should therefore represent a stable stratification. The *u* curve is computed from winds with a northerly component (air temperature below water temperature) and should thus be characteristic for an unstable stratification. The curve *m* was obtained from the entire material. It is seen that the *s* curve reaches a well-developed maximum already at 80 m. above sea level. In case of unstable stratification the eddy-producing layer must evidently be much deeper, since the eddy viscosity does not reach its maximum value within the lowest 200 m.

In this connection it may be of interest to discuss the order of magnitude of the constant a. Denote it

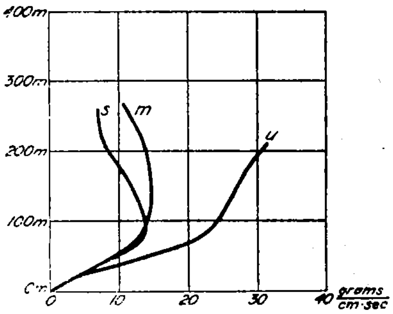


FIG. 6

magn a

We have

$$a=\frac{c_M}{E}.$$

According to the example just given and to measurements of the gustiness we have

$$magn\ E=10^4.$$

On the other hand

$$magn\ c_M=10^1 \text{ to } 10^2.$$

Thus

$$magn\ a=\frac{magn\ c_M}{magn\ E}=\frac{10^1 \text{ to } 10^2}{10^4}=10^{-3} \text{ to } 10^{-2}.$$

Assume for a the value 10^{-3}. We then find from Figure 5 a maximum value for c_M of 43 $\frac{\text{grams}}{\text{cm}\times\text{sec}}$, which is in good agreement with observations.

The constant a being known we are able to calculate numerically the diffusion velocities derived in section II. It is seen that they all fall within the limits which could be drawn *a priori*. Closer comparison is impossible as long as reliable measurements of the eddy diffusion are wanting.

SUMMARY

(1) Starting from an energy equation given by L. F. Richardson, the author derives a differential equation for the atmospheric eddy energy per unit mass (E). The assumption is made that the diffusion of eddies follows the law

$$-c_E\frac{\partial E}{\partial h}\qquad (h=\text{the height above ground}),$$

and that the coefficient c_E, as well as the coefficients of eddy viscosity and eddy convectivity, are proportional to E. The conception *eddy productivity* (ψ) is introduced, and it is shown that the production of eddy energy per unit mass and time is equal to $\psi . E$.

(2) Some integrals to the equation for E are given for the case of an unlimited atmosphere at rest. First, the special stationary distribution of E is derived, which occurs when a narrow eddy-producing layer is surrounded by two eddy-consuming layers. It is shown that even a comparatively high lapse rate of 0.7° C./100 m. is very effective in suppressing eddy diffusion currents.

Furthermore, the diffusion of an initially limited eddy supply during different temperature conditions is studied. It is found that in case of stable stratification

[4] These data will be published in Geografiska Annaler, Stockholm.

the diffusion will never reach beyond certain limits. During unstable stratification the eddies travel with increasing velocity toward infinity.

Finally the equation for E is combined with the equation for eddy convection of heat, and an integral is derived which gives the simultaneous changes in E and θ (the potential temperature) when two infinitely thick adiabatic air layers are separated by a thin superadiabatic layer and overturning sets in. The solution is applied to the discussion of the cause of certain thunderstorms.

(3) Under the assumption that close to the ground the large eddies of the free atmosphere are rapidly annihilated, it is found that E here must be proportional to $\sqrt{h}$.

(4) This boundary condition is used for deriving a stationary distribution of E in a limited atmosphere. Since E and the eddy viscosity are proportional, the resulting curve for E can be compared with known values of the variation of eddy viscosity with height and fair agreement is found.

ACKNOWLEDGMENTS

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A FURTHER STUDY OF EFFECTIVE RAINFALL

551.578.1 : 633

By J. F. VOORHEES

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With every new attempt to find a correlation between rainfall and the growth or yield of some crop one is more deeply impressed with the need for further information concerning effective rainfall and the factors which determine how large a proportion of a given amount of precipitation may be utilized. The problem has been attacked, directly or indirectly, by many investigators and from several angles, but we still have no information that is definite enough to make a satisfactory basis for a correlation between precipitation and yield. Yet we get apparently high correlations in many instances. For example, the correlation coefficient between June rainfall and the yield of oats at Akron, Colo., as calculated by Mr. Mattice, of this bureau, is $+0.91 \pm 0.03$. This is thirty times the probable error but when we try to discover just what this high correlation is good for we find it worth very little indeed. If yields are calculated by the least square formula $y' = bx + a$, the standard deviation of $y - y'$ is found to be 59 per cent less than the standard deviation of y. But the departure of y' from y was 75 per cent of the mean value of y one-fifth of the time, which would make our calculated yield little if any better than a good guess, and practically worthless for predicting yields.

The amount of water used in making a crop of corn has been so often determined and the results are in such close agreement that we may feel reasonably certain that each pound of dry matter in a corn plant will have used from 250 to 400 pounds of water, depending on the fertility of the soil. The larger amount will have been used on the poorer soil. Thus, after it has been harvested and weighed, we can tell about how much water a given crop has used.

It has been shown also that the application of manure or of a straw mulch will increase the moisture-holding capacity of most soils, and measurements have been made of the amount of water available in the upper layers of various soils when saturated. Here, again, we have nothing on which a forecast could be based.

In a previous paper (this REVIEW, February, 1925) the writer attempted to determine the amount of effective rainfall by a process of elimination. First, there was deducted a minimum amount which it was assumed would be lost by immediate evaporation after each rain. Second there was deducted the amount discharged in the streams, as evidently having been of no benefit, whether or not it might have been beneficial under different conditions.

This left an amount which presumably escaped, either by transpiration or evaporation or both. It was suggested that under favorable conditions a growing crop might utilize the major part of this residue and that it might also be able to reduce the portion lost in the streams.

It is now proposed to try to throw a little more light on the question of effective rainfall by considering a particular case.

We first present the charts, A to F, showing for Knoxville for each of the months March to August, inclusive, the total number of days in the 27 years of record which have had rainfalls of the indicated amounts. The abscissæ represent days and the ordinates rainfall in inches and tenths. Each column represents the rainfall for one day and the total of the figures at the bottom gives all the rainy days for the given month for the 27 years.

The heavy horizontal line at the 0.1-inch mark cuts off of the bottom the amount probably lost by immediate evaporation. In March this amounts to 20 per cent of the total rainfall. Then, since the previous study showed that 70 per cent of the March rainfall appeared in the river, a line was drawn at a point cutting off 30 per cent from the bottom and leaving the 70 per cent which appeared in the river above. The part between these two lines, or 10 per cent of the total, is the amount normally used in transpiration or lost later by evaporation

Next, it seemed worth while to make some distinction between surface run-off and water reaching the river by seepage. It was assumed that all water above the minimum stage of the river, for a given month, was due to